Generalized eikonal equation in excitable media

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Numerical simulations show that, in excitable media, the standard eikonal equation describing the dependence of a wave front's local velocity on its curvature fails badly in the presence of significant dispersion [Pertsov *et al.* Phys. Rev. Lett. **78**, 2656 (1997)]. Here we derive a corrected eikonal equation, valid in an unrestricted frequency range, which includes highly dispersive conditions. The derivation, which uses a finite-renormalization technique, is applied to diffusion-reaction equations with generic reactivity functions and two diffusivities of arbitrary ratio. In the important case of equal diffusivities α, we obtain at low curvature, the following contribution to the speed: $[-1+(\omega/c)(\partial c/\partial\omega)](\alpha/r)$, where $1/r$ is the curvature, ω is the frequency, and $c=c(\omega)$ is the speed of a plane wave with that frequency. In the single-diffusivity case there is a further contribution $(\epsilon/c)(\partial c/\partial\epsilon)(\alpha/r)$, where ϵ is the ratio of time scales for diffusing and nondiffusing variables; ϵ is not restricted to a small range. Both cases yield excellent agreement with numerical simulations. Our various formulas are compared with the classical results of Zykov (Biofizika **25**, 888 (1980) [Biophysics **25**, 906 (1980)]) and of Keener [SIAM J. Appl. Math. **46**, 1039 (1986)]. [S1063-651X(97)00706-X]

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INTRODUCTION

Reaction-diffusion systems have been of interest for a long time because of the rich variety of nonlinear wave phenomena that they support. For example, excitable media such as cardiac tissue and Belousov-Zhabotinsky reagents have been intensively studied experimentally, analytically, and computationally with reference to wave propagation, with particular attention to the local speed of the waves. Among the many factors that determine speed, the curvature of the wave front plays a special role. One reason is that, with only a moderate number of simplifying assumptions, we can make definite predictions for the speed C in terms of the curvature $1/r$. Another reason is that the local value of C is related to physiological and chemical observations more directly than many other parameters [1–16].

More than a decade ago, a formula for the propagation speed of a diffusion-reaction wave in terms of its curvature was derived by Zykov [17] in the context of the FitzHugh-Nagumo model with a single diffusion coefficient. His analysis assumed a low-frequency limit, i.e., isolated pulses, thereby eliminating dispersion effects. Furthermore, the ratio ϵ between the short- and long-time scales of the model was taken to be very small, implying wide activation pulses in space or, under a suitable scale change, pulses with sharp fronts. (We refer to these conditions as Zykov's limit.) For small curvature, his eikonal equation reads

$$C=c-\frac{\alpha}{r}, \tag{1}$$

where α is the diffusivity of the medium and $1/r$ is the curvature of the wave front. This equation has been widely used in the literature, for example, in the kinematic theory of spirals [18,19]. It is important, however, to recognize that Eq. (1) can be far from satisfied in real life. In particular, under conditions of high dispersion, the curvature term in Eq. (1) can be wrong by a factor of 2 or more [20]. In what follows we apply a different analytic method (finite renormalization [21]) to the case of concentric circular waves ("target patterns"). We obtain the speed-curvature relation for parameter ranges not covered in earlier research: Dispersion is treated exactly and explicitly, and ϵ need not be restricted to a small range. We shall be dealing with an asymptotic expansion in the curvature of the wave front. To first order in the curvature our results contain, besides the α/r term, significant additions not found in the previous literature. The analytic aspects of our results are contained in Eqs. (33), (37), and (41) of the present paper; these formulas make use of coefficients that must be obtained from plane-wave solutions. When tested on computer simulations, even under conditions of high dispersion, they account accurately for the data; see the brief summary at the end of this article.

The model used here is an unbounded two-dimensional FitzHugh-Nagumo-like medium in which the propagating variables u,v obey the equations

$$\partial_t u-\alpha\nabla^2 u+\Phi_1(u,v)=0, \tag{2}$$

$$\partial_t v-\alpha\delta\nabla^2 v+\delta\Phi_2(u,v)=0, \tag{3}$$

where Φ_1 and Φ_2 are generic reactivity functions; $\alpha>0$ and $\delta>0$ are constant parameters. In order that $\delta\Phi_2$ be any desired function, the form of Φ_2 is considered to be specified only after a value of δ is chosen. In the limiting case $\delta\rightarrow 0$ (single diffusivity) we need to replace Eq. (3) by

$$\partial_t v+\epsilon\Phi_2(u,v)=0, \tag{3$'$}$$

where it is important to note that in our treatment the parameter ϵ is not assumed to be small. The system (2) and (3) with $\delta=1$ is the model of choice for the Belousov-Zhabotinsky reaction [22,23], while system (2),(3′) which closely follows the original FitzHugh-Nagumo model, is applied to electric impulses in the heart muscle [22]. For given δ or ϵ, we consider only such forms of Φ_1 and Φ_2 as will allow the existence of rigidly propagating periodic plane-wave trains.

Our method of derivation can be outlined as follows. For definiteness we think in terms of outward-traveling concentric waves whose chosen constant angular frequency Ω is independent of space and time; a circle of constant u (and constant v) will be referred to as an equipotential. At this point we need to recall that periodic target patterns are not unique, except in the far zone. Such waves are in general not self-sustaining and we therefore postulate that a small but finite disk, concentric with the origin, is periodically stimulated at constant frequency Ω in order to produce the waves. We further assume that, sufficiently far from the origin, the wave form (in time and space) becomes independent of the stimulating wave form, except for its frequency. In the far zone $r\to\infty$, we define a comoving coordinate

$$\rho=\int \mathcal{K}(r)dr-\Omega t, \tag{4}$$

where r is the radial coordinate and t is the time. Equation (4) involves the indefinite integral of an as yet undetermined function $\mathcal{K}$, which may be viewed as the local wave number. For sufficiently large r we consider the variables u,v to depend only on ρ; thus time is ''frozen'' and the comoving feature of ρ is enforced. This constraint will fix the form of $\mathcal{K}$. The wave is now effectively mapped into a periodic plane wave; the mapping process, which we refer to as finite renormalization [21], systematizes and expands Zykov's original procedure [17]. Computing the plane-wave solution is a separate and well-understood problem, not dealt with in the present work. (A notational remark: We make considerable use of wave numbers, k and $\mathcal{K}$, and therefore curvature will be denoted by $1/r$ rather than by K, in contrast to much of existing practice.) The case of inward-traveling waves will be obtainable by changing the sign of r.

Once $\mathcal{K}$ is known, we find the speed-curvature relation by following in time the location of a given equipotential labeled by ρ=const. The time derivative of Eq. (4),

$$\mathcal{K}(r)\frac{dr}{dt}-\Omega=0, \tag{5}$$

gives the radial speed

$$C(r)=\frac{dr}{dt}=\frac{\Omega}{\mathcal{K}(r)}. \tag{6}$$

For $r\to\infty$ we must have a plane wave corresponding to frequency Ω,

$$\mathcal{K}(r)\to k_\Omega=\text{const.} \tag{7}$$

Hence it is reasonable to consider an expansion

$$\mathcal{K}(r)=k_\Omega+\frac{\gamma}{r}+o\left(\frac{1}{r^2}\right), \tag{8}$$

where γ is a constant coefficient. From Eq. (6) we have

$$C(r)=c\left(1-\frac{\gamma}{k_\Omega r}\right)+o\left(\frac{1}{r^2}\right), \tag{9}$$

where $c=C(\infty)=\Omega/k_\Omega$ is the plane-wave speed. This article is mostly concerned with estimates for γ. We do not address the convergence of expansions in powers of $1/r$, in Eq. (8) or in other equations further on.

II. THE METHOD OF FINITE RENORMALIZATION

In this section we concentrate on the system of equations (2),(3). The alternative system (2),(3′) requires an entirely similar technique and will be briefly discussed at the end of Sec. III. We begin by changing variables in the wave equations and subsequently pass to $r\to\infty$. Before making that approximation, we must consider u and v to depend somewhat on r as well as on ρ. Accordingly, let the new independent coordinates be (r,ρ). We set

$$u=u_r(\rho),\quad v=v_r(\rho), \tag{10}$$

functions whose periodicity is

$$u_r(\rho+2\pi)=u_r(\rho),\quad v_r(\rho+2\pi)=v_r(\rho), \tag{11}$$

as can be seen by adding one period to t in Eq. (4). The subscript indicates the residual r dependence, negligible in the region of interest, as we shall show by self-consistency. To the extent that the subscript may be ignored, Eqs. (10) imply an identical temporal wave form everywhere along the radius, apart from a location-dependent phase; in space, all the recurring extrema of the wave are equal to each other and to their values in the corresponding plane wave. These equations also imply that coincident equipotentials of u and v travel together at all times.

Inserting Eqs. (4) and (10) into Eqs. (2) and (3), we have

$$\left[\Omega+\alpha\left(\mathcal{K}'+\frac{\mathcal{K}}{r}\right)\right]u'+\alpha\mathcal{K}^2u''+\alpha\left(\partial_r^2u+\frac{1}{r}\partial_ru+2\mathcal{K}\partial_ru'\right)-\Phi_1=0, \tag{12}$$

$$\left[\Omega+\alpha\delta\left(\mathcal{K}'+\frac{\mathcal{K}}{r}\right)\right]v'+\alpha\delta\mathcal{K}^2v''+\alpha\delta\left(\partial_r^2v+\frac{1}{r}\partial_rv+2\mathcal{K}\partial_rv'\right)-\delta\Phi_2=0, \tag{13}$$

where

$$u'=\frac{\partial u}{\partial\rho},\quad v'=\frac{\partial v}{\partial\rho},\quad \mathcal{K}'=\frac{d\mathcal{K}}{dr}. \tag{14}$$

We express the residual r dependence as an asymptotic series in $1/r$, involving unspecified coefficients $u^{(1)}$, $v^{(1)},\ldots$, which depend only on ρ:

$$\begin{aligned}u_r(\rho)&=u_\infty(\rho)+\frac{1}{r}u^{(1)}(\rho)+\frac{1}{r^2}u^{(2)}(\rho)+\cdots,\\ v_r(\rho)&=v_\infty(\rho)+\frac{1}{r}v^{(1)}(\rho)+\frac{1}{r^2}v^{(2)}(\rho)+\cdots.\end{aligned} \tag{15}$$

We also use Eq. (8) for $\mathcal{K}$. We then find for the ∂_r terms in Eq. (12):

$$\partial_r^2 u + \frac{1}{r}\,\partial_r u + 2\mathcal{K}\partial_r u' = (\text{function of } \rho)\times o\left(\frac{1}{r^2}\right), \tag{16}$$

and similarly for the ∂_r terms in Eq. (13). This article addresses the first correction to plane waves, meaning that Eqs. (12) and (13) need to be enforced through order $1/r$. Therefore, these equations reduce to the ordinary differential equations

$$\left[\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)\right]u' + \alpha\mathcal{K}^2 u'' - \Phi_1 = 0, \tag{17}$$

$$\left[\Omega + \alpha\delta\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)\right]v' + \alpha\delta\mathcal{K}^2 v'' - \delta\Phi_2 = 0, \tag{18}$$

in which the variable r has become a numerical parameter and where we omit the subscript ∞ that occurs in Eq. (15). In what follows, the system (17),(18) will be enforced exactly at first, in order that the method of finite renormalization may be applied to it.

Equations (17) and (18) are to be compared with the plane-wave version of Eqs. (2) and (3), namely,

$$\omega\mathcal{U}' + \alpha k^2\mathcal{U}'' - \Phi_1(\mathcal{U},\mathcal{V}) = 0, \tag{19}$$

$$\omega\mathcal{V}' + \alpha\delta k^2\mathcal{V}'' - \delta\Phi_2(\mathcal{U},\mathcal{V}) = 0, \tag{20}$$

where the single coordinate is $z = kx - \omega t$. Here again, k is the wave number and ω the angular frequency, both constant; therefore, the periodicity is

$$\mathcal{U}(z+2\pi) = \mathcal{U}(z), \quad \mathcal{V}(z+2\pi) = \mathcal{V}(z). \tag{21}$$

Suppose that α and the details of Φ_1, Φ_2 are permanently fixed. Then we consider the system (19),(20), with boundary conditions (21), to be an algorithm that yields k (chosen to be positive) as a function of ω and δ; for uniqueness we require a stable propagating solution. Thus we have

$$k = k(\omega,\delta). \tag{22}$$

We now rewrite Eqs. (17) and (18) in plane-wave form, keeping Eq. (17) unchanged but, following Zykov [17], we multiply Eq. (18) by an overall factor:

$$\left[\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)\right]v' + \frac{\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)}{\Omega + \alpha\delta\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)}\,(\alpha\delta\mathcal{K}^2 v'' - \delta\Phi_2) = 0. \tag{23}$$

The new system (17),(23) is equally obtainable from system (19),(20) under the parameter substitution

$$\omega \to \Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right), \quad \delta \to \frac{\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)}{\Omega + \alpha\delta\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)}\,\delta. \tag{24}$$

The renormalized ω,δ now depend on r through $\mathcal{K}(r)$, but they nevertheless are just numerical parameters in the differential equations for u and v. Furthermore, the term $-\Omega t$ in ρ shows that the periodicity is still given by

$$u(\rho+2\pi) = u(\rho), \quad v(\rho+2\pi) = v(\rho). \tag{25}$$

Therefore, algorithm (22) is still effective, yielding $\mathcal{K}$ instead of k. The desired renormalization condition is

$$\mathcal{K} = k\left(\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right),\quad \frac{\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)}{\Omega + \alpha\delta\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)}\,\delta\right), \tag{26}$$

where the functional form of k is obtained from the plane-wave solutions in a range of ω,δ. If these solutions are considered known, then, for $\delta\neq 0$, Eq. (26) gives the concentric-wave solution in the form of an implicit condition for $\mathcal{K}$.

Are there any special cases that are particularly simple? We note that the second substitution in Eq. (24) becomes an identity, $\delta\to\delta$, in three cases:

$$\mathcal{K}' + \mathcal{K}/r = 0 \quad \text{or} \quad \delta = 0 \quad \text{or} \quad \delta = 1. \tag{27}$$

The first possibility is unphysical since it means that $\mathcal{K}=\text{const}/r$; on the other hand, we see from Eq. (26) that $\mathcal{K}$ is independent of r. Thus we have $\mathcal{K}\equiv 0$ (there are no waves). The next case, $\delta=0$ (single diffusivity), is really a limiting one, where Eq. (3′) must be used and Eq. (26) has to be modified in order to involve ϵ rather than δ, as shown further on. The last case, $\delta=1$ (equal diffusivities), leads to the simplest solution and will be discussed next.

III. SOLVING THE RENORMALIZATION CONDITION

A. Equal diffusivities

When $\delta=1$ we have $\delta\to\delta$ in Eq. (24), so that algorithm (22) and the renormalization condition (26) are reduced to being one dimensional:

$$k = k(\omega), \tag{28}$$

$$\mathcal{K} = k\left(\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)\right). \tag{29}$$

Equation (29) is readily solved in terms of low curvature and unrestricted dispersion. Let us denote by c, $k_\Omega = \Omega/c$, and $dk/d\omega$ the plane-wave quantities evaluated at $\omega=\Omega$. By using the asymptotic expansion (8) we obtain γ. Through order $1/r$, Eq. (29) reads

$$k_\Omega + \frac{\gamma}{r} = k\left(\Omega + \frac{\alpha k_\Omega}{r}\right). \tag{30}$$

With $k_\Omega = k(\Omega) = \Omega/c$, we have for the coefficient of $1/r$

$$\gamma = \frac{\alpha\Omega}{c}\frac{dk}{d\omega}. \tag{31}$$

If, in the plane wave, c is temporarily viewed as a function of ω before being evaluated at $\omega=\Omega$, and with use of $k=\omega/c$, we get from Eq. (31)

$$\gamma = \frac{\alpha\Omega}{c^2}\left(1-\frac{\Omega}{c}\frac{dc}{d\omega}\right). \tag{32}$$

The speed-curvature relation (9) then reads through order $1/r$

$$C(r) = c + \left(-1+\frac{\Omega}{c}\frac{dc}{d\omega}\right)\frac{\alpha}{r}. \tag{33}$$

B. Unequal nonzero diffusivities

To illustrate the two-dimensional procedure, we consider Eq. (26) at low curvature. Again we use Eq. (8) with

$$k_\Omega = k(\Omega,\delta), \tag{34}$$

a known constant obtained from the plane-wave algorithm. Equation (26), through order $1/r$, now reads

$$k_\Omega + \frac{\gamma}{r} = k\left(\Omega + \frac{\alpha k_\Omega}{r}, \quad \delta + \frac{\alpha k_\Omega}{\Omega r}\,\delta(1-\delta)\right). \tag{35}$$

For the coefficient of $1/r$ we find

$$\gamma = \alpha k_\Omega\left(\frac{\partial k}{\partial\omega}\right)_\delta + \frac{\alpha k_\Omega}{\Omega}\,\delta(1-\delta)\left(\frac{\partial k}{\partial\delta}\right)_\omega. \tag{36}$$

Setting $k=\omega/c$ and using Eq. (9), we get the desired result through order $1/r$:

$$C(r) = c + \left[-1+\frac{\Omega}{c}\left(\frac{\partial c}{\partial\omega}\right)_\delta + \frac{\delta(1-\delta)}{c}\left(\frac{\partial c}{\partial\delta}\right)_\omega\right]\frac{\alpha}{r}, \tag{37}$$

where all coefficients, including c itself, are the plane-wave versions evaluated at frequency Ω and parameter δ. The subscript δ or ω refers to the parameter that is being kept constant. The above result has Eq. (33) as a special case.

C. Single diffusivity

We finally go over to the single-diffusivity case $\delta\to 0$, where we use Eq. (3′) rather than Eq. (3), so that we must modify Eqs. (24) and (26). The procedure is the same as in the $\delta\neq 0$ case, but here the independent plane-wave parameters are ω and ϵ, giving rise to the plane-wave algorithm

$$k = k(\omega,\epsilon). \tag{38}$$

[This ‘‘recycles’’ the notation of Eq. (22); k is now a different function.] The renormalization condition reads

$$\mathcal{K} = k\left(\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right), \quad \left[\Omega + \alpha\left(\mathcal{K}' + \frac{\mathcal{K}}{r}\right)\right]\frac{\epsilon}{\Omega}\right), \tag{39}$$

which is readily solved for the term of order $1/r$. Using Eq. (8), we have for the coefficient of that term

$$\gamma = \alpha k_\Omega\left(\frac{\partial k}{\partial\omega}\right)_\epsilon + \frac{\alpha k_\Omega \epsilon}{\Omega}\left(\frac{\partial k}{\partial\epsilon}\right)_\omega. \tag{40}$$

Through order $1/r$, Eq. (9) gives, with use of $k=\omega/c$,

$$C(r) = c + \left[-1+\frac{\Omega}{c}\left(\frac{\partial c}{\partial\omega}\right)_\epsilon + \frac{\epsilon}{c}\left(\frac{\partial c}{\partial\epsilon}\right)_\omega\right]\frac{\alpha}{r}, \tag{41}$$

where the ingredients are the plane-wave quantities measured at $\omega=\Omega$. For $\epsilon\to 0$ and fixed Φ_2, Eq. (41) coincides formally with Eq. (33)—a misleading resemblance. Indeed, at any given nonzero frequency, the value of ϵ cannot in general be made arbitrarily small, as a plausibility argument will indicate. Let ω be assumed fixed. Then, as $\epsilon\to 0$, the ever-increasing pulse width causes each front and the preceding tail to approach one another. This will lower the frequency, in contradiction with our assumption. Nevertheless, we can correctly take $\omega\to 0$ *before* $\epsilon\to 0$, thus arriving at the Zykov limit. Our numerical checks of Eq. (41) show some interesting features that are probably related to the nonuniformity of this double limit, as we explain at the end of Sec. IV.

IV. DISCUSSION

In the preceding section we have derived the speed-curvature formulas (33), (37), and (41), valid for equal diffusivities, unequal nonzero diffusivities, and single diffusivity respectively. A negative r will fit concave (converging) fronts. The coefficients in the formulas are taken from a solution of the plane-wave problem. In particular, they involve the plane-wave dispersion curve and the dependence of the plane-wave speed on a combination of the diffusivity ratio and the ratio of time constants. The following assumptions have been made: (a) a generic two-dimensional uniform isotropic FitzHugh-Nagumo medium, (b) concentric waves with a given frequency Ω, and (c) a sufficient distance from the center. No specific assumptions are needed concerning the reactivity functions or the dispersion; nor are any assumptions made about the wave being sparse or having a very small ratio of time constants. The low-curvature assumption (c) is inherent to the phenomenon rather than to the mathematics. Since the excitation pulses at the center are to some extent arbitrary wave forms in time, the wave will not even be unique unless measured at a sufficient distance from the center. How far is sufficient? We have no complete analytic answer, but Eqs. (15) and (16) lead us to expect that, although the transients themselves may decay as slowly as $o(1/r)$, their effect on the wave speed is not more than $o(1/r^2)$, which is consistent with a speed formula that goes through $o(1/r)$.

Excellent confirmation of our results by numerical calculations is demonstrated in [20]. As far as their overlap with the analytic work of Zykov and of Keener are concerned, it can be summarized as follows (within the context of low curvature). Zykov [17] assumes a single diffusivity; he also requires waves that are (i) in uniform rigid translational motion; (ii) of low frequency $\Omega\to 0$, which essentially implies solitary pulses, thereby avoiding dispersion problems; and (iii) characterized by sharp fronts on the appropriate scale, i.e., the parameter ϵ is small. Under those conditions, and if the limits $\Omega \to 0$ and $\epsilon \to 0$ are taken in that order, our formula (41) agrees with Zykov's Eq. (1). Here the distinction between translational and concentric appears to be unimportant. In conclusion, we see no overlap between our results and Eq. (1) except under Zykov's limiting procedure. Keener's speed-curvature equation [24,25] essentially retains only one of Zykov's restrictions, namely (iii), the smallness of ϵ. It is, however, too implicit for ready comparison with our formulas in their domain of validity. It reads (as adapted to our notation)

$$C(r) = c(v_f) - \frac{\alpha}{r}, \tag{42}$$

where dispersive effects are taken into account by the term $c(v_f)$. This represents the speed of a plane wave whose frequency is not necessarily Ω, but is adjusted so that the variable v (nearly constant as we intersect the steep fronts of u) has the same value v_f as in the actual fronts under consideration. A comparison with our single-diffusivity formula (41) in the case $\epsilon \to 0$ indicates that the term $c(v_f)$ has itself an implicit first-order dependence on the curvature $1/r$. The same conclusion must apply to the other cases, covered by our Eqs. (33) and (37).

The formulas can be rewritten without derivatives, giving rise to an alternative procedure for numerical verification and at the same time making their appearance more suggestive of Keener's result (42), although not formally identical to it. We define a function $c(\omega)$, $c(\omega,\delta)$, or $c(\omega,\epsilon)$, which represents the speed of a plane wave in the cases corresponding to Eqs. (28), (22), and (38), respectively. We can then verify directly (through a Taylor-series expansion in $1/r$) that the following formulas reproduce our results to $o(1/r)$: Eq. (33),

$$C(r) = c\left(\left(1 + \frac{\alpha}{cr}\right)\Omega\right) - \frac{\alpha}{r}; \tag{43}$$

Eq. (37),

$$C(r) = c\left(\left(1 + \frac{\alpha}{cr}\right)\Omega, \left[1 + \frac{(1-\delta)\alpha}{cr}\right]\delta\right) - \frac{\alpha}{r}; \tag{44}$$

and Eq. (41),

$$C(r) = c\left(\left(1 + \frac{\alpha}{cr}\right)\Omega, \left(1 + \frac{\alpha}{cr}\right)\epsilon\right) - \frac{\alpha}{r}. \tag{45}$$

In formulas (43)–(45) above, the right-hand sides are explicit expressions, provided we have determined the functional dependence of c for plane waves.

A comment still needs to be made about the somewhat unconventional parametrization employed in Eq. (3). Everything done here could equally well be done, although in a more unwieldy fashion, with the term $\delta\Phi_2(u,v)$ denoted by $\epsilon\Phi_2(u,v)$. The two-dimensional parameter space (ω,δ) would expand to three dimensions (ω,δ,ϵ) and the speed-curvature relations in the version corresponding to Eqs. (33), (37), and (41) would involve one more partial derivative. For completeness we present the formulas obtained in that way. The functional dependence of c is now $c = c(\omega,\delta,\epsilon)$. Corresponding to Eq. (37), we have the equivalent formula

$$C(r) = c + \left[-1 + \frac{\Omega}{c}\left(\frac{\partial c}{\partial \omega}\right)_{\delta,\epsilon} + \frac{\delta(1-\delta)}{c}\left(\frac{\partial c}{\partial \delta}\right)_{\omega,\epsilon} + \frac{\epsilon(1-\delta)}{c}\left(\frac{\partial c}{\partial \epsilon}\right)_{\omega,\delta}\right]\frac{\alpha}{r} \tag{46}$$

and, equivalently to Eq. (44) through $o(1/r)$,

$$C(r) = c\left(\left(1 + \frac{\alpha}{cr}\right)\Omega, \left[1 + \frac{(1-\delta)\alpha}{cr}\right]\delta, \left[1 + \frac{(1-\delta)\alpha}{cr}\right]\epsilon\right) - \frac{\alpha}{r}. \tag{47}$$

How well do our results agree with numerical data? In a separate study [20] we have tested two series of simulations: (a) the system (2),(3) with $\delta = 1$ (two equal diffusivities) and (b) the system (2),(3′) (single diffusivity). We chose a parametrization and a set of frequencies that would confront the eikonal relation with a high dispersion in at least a portion of the parameter range. We kept ϵ very small, as is done in the existing literature, in order to single out the effect of dispersion. In simulations (a), we had good quantitative agreement with Eq. (37). Simulations (b), intended as a test of the $\partial c/\partial\omega$ term in Eq. (41), actually demonstrated that the $\partial c/\partial\epsilon$ term could be fully comparable in size to the $\partial c/\partial\omega$ term, in spite of the small ϵ. (This curious feature fits in with the comments at the end of Sec. III.) The test therefore involved both terms and again gave excellent agreement between theory and simulation. In all cases tested and within the accuracy of our figures, the actual curvature term exceeded its Zykov limit, sometimes by over 100%.

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